Available online at www.sciencedirect.com
SCIENCE DIRECT®
ELSEVIER
International Journal of Solids and Structures 42 (2005) 1271–1285
INTERNATIONAL JOURNAL OF SOLIDS and STRUCTURES
www.elsevier.com/locate/ijsolstr

Effective approach to the contact problem for a stratum

P. Malits *

RCQCE at Department of Communication Engineering, Holon Academic Institute of Technology, 52 Golomb Street, Holon, 58102 Israel

Received 9 August 2004
Available online 13 October 2004

Abstract

A novel effective algorithm for the problem of the circular punch in contact with a stratum rested on a rigid base is suggested in this paper. The problem is reduced to the Fredholm integral equations of the second kind. In contrast to the Cooke–Lebedev method and the moments method, which are traditionally employed, the operators of these integral equations are strictly positive definite even in the limiting case of the zero thickness. The latter provides efficient applications of numerical methods. It is also shown that a special approximation enables to obtain an approximate solution via a finite system of linear algebraic equations. As example, the well-known problem for a homogeneous layer is studied. An approximate analytical solution is found with a certain iterative method for a flat punch. This solution is remarkable accurate and possesses the right asymptotic behavior for both a very thin and a very thick layers. Asymptotic formulas for the thin inhomogeneous stratum indented by an indenter of arbitrary profile are pointed out.

Keywords: Contact problem; Inhomogeneous stratum; Dual integral equations; Fredholm equation; Asymptotic expansion

1. Introduction

In the present investigation, we consider an elastic isotropic inhomogeneous (or stratified) medium consisting of layers with parallel boundaries which is indented by a frictionless circular punch. The elastic properties of the layers may vary in the depth direction only. The considered stratum is bonded to the rigid base or can slip on the rigid substrate without friction.

There are many publications in the treated field. In particular, a number of the papers are dedicated to the simplest case of a homogeneous layer. This case is covered comprehensively in the books by Ufliand (1967) and Vorovich et al. (1974). The general case also was treated intensively. It is partly covered in the book by Shevlyakov (1977).

* Address: Hanaviim Street 12/3, Ashdod 77473, Israel. Tel.: +967088647194; fax: +967035026685.
E-mail address: malitsp@hotmail.co.il

0020-7683/$ - see front matter
doi:10.1016/j.ijsolstr.2004.08.008

Two approaches were employed for solving the studied problem. In the first of them, the problem is formulated in the form of the dual integral equations involving Bessel function of the first kind:

$$R\int_0^\infty A_n(p)f(p)J_{|n|}(pr)\,\mathrm{d}p = u_n(r), \quad 0 \leqslant r \leqslant 1, \tag{1}$$

$$\int_0^\infty pA_n(p)J_{|n|}(pr)\,\mathrm{d}p = 0, \quad 1 \leqslant r < \infty. \tag{2}$$

The unknown $A_n(p)$ is the Hankel transform of the coefficient $p_n(r)$ of the nth harmonic in the Fourier expansion of the contact pressure $\sum p_n(r)\exp(\mathrm{i}n\theta)$, R is the radius of the contact zone, (r,θ,z) are the dimensionless cylindrical coordinates, $J_n(pr)$ is the Bessel function of the first kind.

The conditions of the punch equilibrium are expressed directly via $A_0(p)$, $A_1(p)$ and the arm d of the imbedding force $\mathscr{P}$

$$2\pi R^2 A_0(0) = \mathscr{P}, \tag{3}$$

$$4\pi R^3 \lim_{p\to 0}\frac{\mathbf{Re}[A_1(p)]}{p} = \mathscr{P}d. \tag{4}$$

The odd function $f(p)$ depends on the mechanical and geometrical characteristics: the shear moduli $G(z)$, the Poisson's ratios $v(z)$ and the thickness. It is positive for $p > 0$ and has the following asymptotic behavior:

$$f(p) = \frac{\pi\gamma}{2}p + O(p^3), \quad \gamma > 0, \text{ as } p \to 0, \tag{5}$$

$$f(p) = 1 + l(p) \text{ as } p \to \infty, \tag{6}$$

where $l(p) = O(p^k\exp(-bp))$, $k \geqslant 0$, $b > 0$ if every layer is homogeneous, and $l(p) = O(1/p)$ in the general case; the positive number γ becomes $O(\lambda)$ as the dimensionless thickness of the stratum λ approaches zero, except the case of the incompressible layer bonded to the rigid base. This asymptotic behavior is well-known. The explicit expressions of the coefficient γ for the piecewise homogeneous stratum, which also take into account a stratification, arise from the results by Privarnikov and Shevlyakov and their co-workers (Shevlyakov, 1977). A piecewise constant approximation of $G(z)$ and $v(z)$ yields in the limit the value of γ for arbitrary inhomogeneity.

For the non-stratified inhomogeneous layer which is bonded to the rigid substrate,

$$\gamma = \frac{G_0}{\pi(1-v_0)}\int_0^\lambda \frac{1-2v(z)}{(1-v(z))G(z)}\mathrm{d}z, \tag{7}$$

where G_0 and v_0 are the values of the elastic constants in the contact plane (Malits, 2004). We see that in this special case $\gamma = 0$ for an incompressible layer.

The given functions $u_n(r)$ in the right-hand are the Fourier coefficients of the function: $u(r,\theta) = G_0[c + c_1 Rr\cos\theta - v(r,\theta)]/(1-v_0)$, where $z = v(r,\theta)$ is the equation of the punch surface, $c + c_1\cos\theta$ is a rigid displacement of the punch.

The dual integral equations (1) and (2) are reduced with the Cooke–Lebedev method (Cooke, 1956; Lebedev, 1957) to the Fredholm integral equations of the second kind whose solutions may be found with numerical methods (Ufliand, 1967; Shevlyakov, 1977).

The second approach is based on the direct treatment of the equivalent Fredholm integral equations of the first kind for the contact pressure

$$R\int_0^1 p_n(t)\left(\int_0^\infty f(p)J_n(pr)J_n(pt)\,\mathrm{d}p\right)t\,\mathrm{d}t = u_n(r), \quad 0 \leqslant r \leqslant 1, \tag{8}$$

with the moment method by using $P^n_{2k+n}(\sqrt{1-r^2})/\sqrt{1-r^2}$, $P^n_k(x)$ is the associated Legendre polynomial, as the basic functions that is referred as "the orthogonal polynomials method". This choose is rested on the fact that these basic functions are the eigenfunctions of (8) in the special case of the homogeneous half-space when $f(p)=1$. These results may be derived as some realization of the Cooke–Lebedev method as well (Shevlyakov, 1977).

It is appeared even in the simplest case of the homogeneous layer that the methods mentioned above fail as the thickness of the layer λ is very small. The explanation is that $f(p)\to 0$ as $\lambda\to 0$, and, consequently, the kernels of the integral equations (8) become small meanwhile the right-hand parts $u_n(r)$ are not depend on λ. In other words, the equations are ill-conditioned. In order to overcome this difficulties Alexandrov and Babeshko (Vorovich et al., 1974) suggested another approach which reduced (8) to the infinite system of the linear algebraic equations. Their method based on the Wiener–Hopf technique requires knowledge of the complex zeroes of $f(p)$ or, at least, its special approximations. This approach is convenient to derive a leading asymptotic term as $\lambda\to 0$, but it is complicated for application in the general case.

The problem for the inhomogeneous or stratified medium is much more complicated because of the tangle dependence of $f(p)$ on the mechanical and geometrical characteristics as well as the necessity to calculate this function numerically. The theories which have been derived from the physical arguments (Barber, 1990; Jaffar, 1989) cannot be used for the inhomogeneous stratum as well. It is important, in this situation, to have a simple stable algorithm which conforms to the specificity of the problem. We suggest in this paper a certain novel approach and demonstrate its virtues on the "classic" problem for the homogeneous layer indented by a flat punch. In particular, we derive regular integral equations whose operators remain strictly positive definite in the limit $\lambda\to 0$ and discuss their approximate solution. For the homogeneous layer, this enables us to determine a solution for any thickness and to point out simple approximate formulas. Comparisons with the known results of numerical calculations manifest the remarkable accuracy of these formulas for all values of the thickness λ. In the case of the thin inhomogeneous stratum, leading asymptotic terms are pointed out.

It is well known that a solution of the dual integral equations for any integer n can be found if the algorithm for $n=0$ is available. Therefore we will consider further the axially symmetric strain state: $n=0$, and another important case: $n=1$, only.

The dual integral equations of the treated type also often arise in connection with application Hankel transforms to mixed boundary problems in various branches of mechanics, such as fracture mechanics, dynamic contact problems, mechanics of piezoelectric medium, hydrodynamics, and so forth. The author hopes that the method of this paper will find wide applications. The rigorous mathematical theory of the more general integral equations for arbitrary integer index n will be published in the nearest future.

2. Axially symmetric problem

The starting point is the discontinuous integral

$$\int_0^\infty p\chi^\gamma_{1,1}(p,t)J_0(pr)\,\mathrm{d}p=\begin{cases}\dfrac{2(t^2+\gamma^2-r^2)}{\pi\gamma t\sqrt{(t^2-(r+\gamma)^2)(t^2-(r-\gamma)^2)}}, & 0<r<t-\gamma,\\ =0, & r>t-\gamma,\end{cases}\tag{9}$$

where $\chi^\gamma_{\mu,\nu}(p,t)=Y_\nu(pt)J_\mu(p\gamma)-Y_\mu(p\gamma)J_\nu(pt)$; $J_\mu(pt)$, $Y_\nu(pt)$ are Bessel functions of the first and second kind, respectively, and the parameter γ is defined by (5). This integral was evaluated by means of the known integral involving three Bessel functions (Prudnikov et al., 1986). Application the inversion formula for the Hankel transform and the following operation $\frac{1}{t}\frac{\mathrm{d}}{\mathrm{d}t}t$ give us the operator transforming $J_0(pr)$ into $p\chi^\gamma_{1,0}(p,t)$

$$\mathbf{R}_0[J_0(pr)] = p\chi_{1,0}^{\gamma}(p,t), \tag{10}$$

$$\mathbf{R}_0[\cdot] = \frac{2}{\pi\gamma t}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^{t-\gamma}(\cdot)\frac{(t^2+\gamma^2-r^2)r\mathrm{d}r}{\sqrt{(t^2-(r+\gamma)^2)(t^2-(r-\gamma)^2)}}. \tag{11}$$

Using the operator $\mathbf{R}_0$ and (10) allows us to transform the first of the dual equations (1) to

$$R\int_0^{\infty} pA_0(p)f(p)\chi_{1,0}^{\gamma}(p,t)\,\mathrm{d}p = \mathbf{R}_0[u_0(r)], \quad \gamma \leqslant t \leqslant 1+\gamma. \tag{12}$$

Now we seek $A_0(p)$ of the form

$$A_0(p) = \frac{\pi\gamma}{2R}p\int_{\gamma}^{1+\gamma} s\omega(s)\chi_{1,0}^{\gamma}(p,s)\,\mathrm{d}s, \tag{13}$$

where $\omega(s)$ is an auxiliary function. Integrating by parts, one can rewrite this representation as

$$\frac{2R}{\pi\gamma}A_0(p) = (1+\gamma)\omega(1+\gamma)\chi_{1,1}^{\gamma}(p,1+\gamma) - \int_{\gamma}^{1+\gamma} s\chi_{1,1}^{\gamma}(p,s)\,\mathrm{d}\omega(s). \tag{14}$$

Putting (14) in the second of the dual equations (2) and interchanging the order of integration, we ascertain by means of the integral (9) that this equation is satisfied identically. Substituting (13) to (12) and taking into account the inversion formula for the Weber–Orr transform (Titchmarsh, 1924)

$$\begin{aligned}\varpi(t) &= \int_0^{\infty}\frac{p\overline{\varpi}(p)\chi_{1,v}^{\gamma}(p,t)}{J_1^2(p\gamma)+Y_1^2(p\gamma)}\mathrm{d}p, \ \ v=1,2,\\ \overline{\varpi}(p) &= \int_{\gamma}^{\infty} s\varpi(s)\chi_{1,v}^{\gamma}(p,t)\,\mathrm{d}s,\end{aligned} \tag{15}$$

we obtain the Fredholm integral equation of the second kind

$$(\mathbf{I}+\mathbf{K})\omega = g(t), \ \ \gamma \leqslant t \leqslant 1+\gamma, \tag{16}$$

$$\mathbf{K}\omega = \int_{\gamma}^{1+\gamma} s\omega(s)K(t,s)\mathrm{d}s, \ \ g(t) = \mathbf{R}_0[u_0(r)], \tag{17}$$

$$K(t,s) = \int_0^{\infty} p[L(p)-1]\frac{\chi_{1,0}^{\gamma}(p,t)\chi_{1,0}^{\gamma}(p,s)}{J_1^2(p\gamma)+Y_1^2(p\gamma)}\mathrm{d}p, \tag{18}$$

where $L(p) = \pi\gamma p f(p)[J_1^2(p\gamma)+Y_1^2(p\gamma)]/2$, $L(p) = 1+O(p^2\ln p)$ as $p\to 0$, $L(p) = 1+O(p^{-c})$ as $p \gg 1$. The symmetric kernel $K(t,s)$ is continuous when the elastic properties are piecewise constant ($c=\infty$), or possesses a logarithmic singularity in the other case ($c=1$).

Upon inverting the Hankel transform we establish from (14) a simple formula for the contact pressure

$$Rp_0(r) = \frac{(1+2\gamma+2\gamma^2-r^2)\omega(1+\gamma)}{\sqrt{(1-r^2)[(1+2\gamma)^2-r^2]}} - \int_{r+\gamma}^{1+\gamma}\frac{(s^2+\gamma^2-r^2)\mathrm{d}\omega(s)}{\sqrt{(s^2-(r+\gamma)^2)(s^2-(r-\gamma)^2)}}. \tag{19}$$

Hence the stress concentration in the vicinity of the punch edge is proportional to the value of the auxiliary function

$$\lim_{r\to 1} p_0(r)\sqrt{1-r^2} = \frac{\sqrt{\gamma+\gamma^2}}{R}\omega(1+\gamma). \tag{20}$$

The expression for the embedding force follows from (3) and (13)

$$\mathscr{P} = 2\pi R \int_{\gamma}^{1+\gamma} s\omega(s)\,\mathrm{d}s. \tag{21}$$

Another very simple relation can be derived for the work W done by the contact pressure $p_0(r)$ in causing the punch displacement $u_0(r)$. Introducing the inner product of two functions as

$$(\psi(s), \varphi(s)) = \int_{\gamma}^{1+\gamma} s\psi(s)\varphi(s)\,\mathrm{d}s, \tag{22}$$

we have

$$\begin{aligned}\frac{G_0}{1-\nu_0}W &= 2\pi R^2(p_0(r), u_0(r)) = 2\pi R^3\int_0^\infty f(p)A_0^2(p)\,\mathrm{d}p = \pi^2\gamma R\int_0^\infty \frac{pL(p)\overline{\varpi}^2(p)}{J_1^2(p\gamma)+Y_1^2(p\gamma)}\,\mathrm{d}p\\ &= \pi^2\gamma R(\omega(s), (\mathbf{I}+\mathbf{K})\omega).\end{aligned} \tag{23}$$

Then

$$W = \frac{\pi^2\gamma(1-\nu_0)R}{G_0}\int_{\gamma}^{1+\gamma} s\omega(s)g(s)\,\mathrm{d}s. \tag{24}$$

To study the integral equation, we will use the Parseval theorem for the Weber transforms

$$\int_0^\infty \frac{p\overline{\varpi}^2(p)}{J_1^2(p\gamma)+Y_1^2(p\gamma)}\,\mathrm{d}p = \int_{\gamma}^{\infty} t\varpi^2(t)\,\mathrm{d}t \tag{25}$$

and the Hilbert space induced by the inner product (22). It is readily seen from (23) that

$$m\|\omega\|^2 \leqslant \frac{G_0}{\pi^2(1-\nu_0)\gamma R}W \leqslant M\|\omega\|^2, \tag{26}$$

$$m = \inf L(p) > 0, \quad M = \sup L(p) < \infty. \tag{27}$$

This manifests that $\sqrt{(\omega(s), (\mathbf{I}+\mathbf{K})\omega)}$ is an equivalent norm. The convergence in the functional space defined by this norm is equivalent to the convergence in our Gilbert space and means the convergence by the energy of the treated mechanical system. The Schwartz inequality, applied to (24), yields the upper estimate for the error

$$|W - \widetilde{W}| \leqslant \frac{\pi^2(1-\nu_0)\gamma R}{G_0}\|\omega(s) - \widetilde{\omega}(s)\|\sqrt{\int_{\gamma}^{1+\gamma} sg^2(s)\,\mathrm{d}s}, \tag{28}$$

where $\widetilde{\omega}(s)$ is some approximate solution and $\widetilde{W}$ is the corresponding approximate value of the punch energy.

The estimate (26) implies the positive definiteness of the operator $\mathbf{I}+\mathbf{K}$ and provides the existence of an unique solution. The stated allows to determine a solution with both projective and iterative methods. These methods are effective as: (1) $\inf L(p)$ is not approach zero; (2) $\sup L(p)$ is not too large. The first of these requirements is expected to be fulfilled. The second may be violated if one of the lower layers is weak or stratum is bonded to the base and ν_0 approaches 1/2.

The spectrum of the self-adjoint strictly positive definite operator $\mathbf{B} = \mathbf{I}+\mathbf{K}$ is localized within the interval $[m, M]$. Then a solution can be determined with the following iterative method (see Krasnosel'skii et al., 1972)

$$\omega_{k+1}(t) = \frac{T_i\left(\frac{2\mathbf{B}-M-m}{M+m}\right)}{T_i\left(\frac{M+m}{m-M}\right)}\omega_k(t) - \mathbf{B}^{-1}\left[\frac{T_i\left(\frac{2\mathbf{B}-M-m}{M+m}\right)}{T_i\left(\frac{M+m}{m-M}\right)} - \mathbf{I}\right]g(t), \tag{29}$$

where $T_i(x)$ is the Chebyshev polynomial, $i \geqslant 1$. This method converges as the geometric progression with the quotient

$$q = \left\| \frac{T_i\left(\frac{2\mathbf{B}-M-m}{M+m}\right)}{T_i\left(\frac{M+m}{m-M}\right)} \right\| \leqslant \frac{1}{\left|T_i\left(\frac{M+m}{m-M}\right)\right|}.$$

In order to use simple iterations, we should know the norm of the operator $\mathbf{K}$. The rough estimate of this norm can be established by employment of the Parseval theorem (25)

$$\|\mathbf{K}\| = \sup_{\|\varpi\|=1} |(\varpi, \mathbf{K}\varpi)| = \sup_{\|\varpi\|=1} \left| \int_0^\infty \frac{p[L(p)-1]\overline{\varpi}^2(p)}{J_\nu^2(p\lambda)+Y_\nu^2(p\lambda)}\,\mathrm{d}p \right| \leqslant \sup|L(p)-1|. \tag{30}$$

If $\sup|L(p)-1| < 1$, the iterations converge. This quantity may be expected to be small for many cases since $L(p)-1 = 0$ at points $p = 0$ and $p = \infty$. It should be mentioned, however, that the method (29) is preferable because of its rapid convergence.

Another approximate solution is based on the special approximation of the function $L(p)$:

$$L(p) \approx \widetilde{L}(p) = \prod_{m=1}^{N} \frac{p^2+a_m^2}{p^2+b_m^2}, \quad \widetilde{L}(0) \approx 1, \tag{31}$$

where a_m and b_m are distinct positive numbers. The operator of the Fredholm integral equation of the second kind (16) remains strictly positive definite under this approximation. It can be rewritten as the dual integral equations

$$\int_0^\infty \overline{\varpi}(p) \prod_{m=1}^{N} \frac{p^2+a_m^2}{p^2+b_m^2} \frac{p\chi_{1,0}^{\gamma}(p,t)}{J_1^2(p\gamma)+Y_1^2(p\gamma)}\,\mathrm{d}p = g(t), \quad \gamma \leqslant t \leqslant 1+\gamma, \tag{32}$$

$$\int_0^\infty \overline{\varpi}(p) \frac{p\chi_{1,0}^{\gamma}(p,t)}{J_1^2(p\gamma)+Y_1^2(p\gamma)}\,\mathrm{d}p = 0, \quad 1+\gamma \leqslant t < \infty, \tag{33}$$

with $\overline{\varpi}(p) = 2RA_0(p)/\pi\gamma p$.

The solution $\overline{\varpi}(p)$ is sought of the form

$$\overline{\varpi}(p) = \frac{1}{\widetilde{L}(p)} \int_\gamma^{1+\gamma} s\varphi(s)\chi_{1,0}^{\gamma}(p,s)\,\mathrm{d}s - \frac{J_l(p)}{p^{l+1}} \sum_{k=1}^{N} \frac{x_k}{p^2+a_k^2}, \tag{34}$$

where $\varphi(s)$ is a new unknown function, x_k are some undetermined numbers and l is any positive number.

Substituting (34) to (32), we find

$$\varphi(t) = g(t) + \sum_{m=0}^{N} x_m\varphi_m(t), \tag{35}$$

$$\varphi_m(t) = \int_0^\infty \frac{p^{-l}\widetilde{L}(p)J_l(p)\chi_{1,0}^{\gamma}(p,t)}{(p^2+a_m^2)[J_1^2(p\gamma)+Y_1^2(p\gamma)]}\,\mathrm{d}p. \tag{36}$$

After evaluation arising integrals by contour integration, the second equation (33) becomes

$$\sum_{k=0}^{N}\left[x_k\frac{I_l(a_k)}{a_k^{l+1}}-\Pi_k\int_{\gamma}^{1+\gamma}s\varphi(s)W_{k,0}(s)\,\mathrm{d}s\right]\frac{a_kK_0(a_kt)}{K_1(a_k\gamma)}=0,\quad t>1+\gamma, \tag{37}$$

where $W_{k,v}(s)=I_\mu(a_k\ {}_s)K_1(a_k\gamma)-(-1)^{1-\mu}\ I_1(a_k\gamma)K_\mu(a_k\ {}_s)$, $I_\mu(t)$ and $K_\mu(t)$ are the modified Bessel functions;

$$\Pi_k=\frac{2}{\pi}\prod_{m=1}^{N}(b_m^2-a_k^2)\prod_{\substack{m=1\\m\neq k}}^{N}(a_m^2-a_k^2)^{-1}.$$

The integral representation

$$K_0(at)=\int_t^\infty\frac{\exp(-as)}{\sqrt{s^2-t^2}}\mathrm{d}s$$

and the linear independence of the exponents show the functions $K_0(a_kt)$ to be linearly independent as well. Thus the coefficients in (37) are equal to zero. Then putting (35) into these coefficients leads to the system of the linear algebraic equations

$$\sum_{m=1}^{N}x_ma_{m,k}=-g_k,\quad k=1,2,\ldots,N,\quad g_k=\int_\gamma^{1+\gamma}sg(s)W_{k,0}(s)\,\mathrm{d}s, \tag{38}$$

$$a_{m,k}=\int_\gamma^{1+\gamma}s\varphi_m(s)W_{k,0}(s)\,\mathrm{d}s-\frac{I_l(a_k)}{\Pi_ka_k^{l+1}}\delta_{k,m}. \tag{39}$$

Here $\delta_{k,m}$ is the Kronecker delta. This system is equivalent to the integral equation with the strictly positive definite operator and, therefore, has an unique solution.

The matrix elements $a_{m,k}$ can be evaluated explicitly. We replace $\varphi_m(s)$ by its integral representation (36) and interchange the order of integration. The inner integral is evaluated by means of the known integral (Prudnikov et al., 1986). Then the arising integral can be readily found by examination of the contour integrals

$$\oint_{\mathfrak{L}}\frac{z^{s-l}\widetilde{L}(z)J_l(z)H_s^{(1)}(z(1+\gamma))}{(z^2+a_m^2)(z^2+a_k^2)H_1^{(1)}(z\gamma)}\mathrm{d}z,$$

where $s=0,1$; the contour $\mathfrak{L}$ consists of the interval $[-R,R]$ of the real axis and the arc $z=R$, $0\leqslant\arg(z)\leqslant\pi$; $H_s^{(1)}(z)$ is the Bessel function of the third kind.

Finally, we have

$$a_{m,k}=\frac{\pi}{2}\sum_{r=1}^{N}\frac{(1+\gamma)U_{r,k}I_l(b_r)b_r^{-l-1}\prod\limits_{l=1}^{N}(b_r^2-a_l^2)}{(b_r^2-a_k^2)(b_r^2-a_m^2)K_1(b_r\gamma)\prod\limits_{\substack{l=1\\l\neq r}}^{N}(b_r^2-b_l^2)^{-1}}, \tag{40}$$

$$U_{r,k}=b_rK_1(b_r(1+\gamma))W_{k,0}(1+\gamma)+a_kK_0(b_r(1+\gamma))W_{k,1}(1+\gamma).$$

We shall prove that if the approximation is fairly accurate

$$\begin{aligned}&\frac{|\varepsilon(p)|}{L(p)} \leqslant \varepsilon, \quad \varepsilon(p) = L(p) - \widetilde{L}(p),\\&\frac{|\varepsilon(p)|}{\widetilde{L}(p)} = \frac{|\varepsilon(p)|}{L(p)[1-\varepsilon(p)/L(p)]} \leqslant \frac{\varepsilon}{1-\varepsilon},\end{aligned} \tag{41}$$

then the corresponding approximative value of the punch energy $\widetilde{W}$ is close to the exact value W.

Let $\omega_0(t)$ be a solution of the equation $(\mathbf{I}+\widetilde{\mathbf{K}})\omega = g(t)$ where $\widetilde{\mathbf{K}}$ is obtained from $\mathbf{K}$ by replacing $L(p)$ by $\widetilde{L}(p)$. We write

$$(\mathbf{I}+\mathbf{K})(\omega-\omega_0) = \int_0^\infty p\frac{\varepsilon(p)\overline{\omega}_0(p)\chi_{1,0}^{\gamma}(p,s)}{J_1^2(p\gamma)+Y_1^2(p\gamma)}\,\mathrm{d}p.$$

Since the operator $\mathbf{I}+\mathbf{K}$ is symmetric, we obtain

$$\begin{aligned}\frac{G_0}{\pi^2(1-\nu_0)\gamma R}|W-\widetilde{W}| &= \int_\gamma^{1+\gamma} t(\omega(t)-\omega_0(t))g(t)\,\mathrm{d}t = (\omega(t)-\omega_0(t),(\mathbf{I}+\mathbf{K})\omega)\\&= (\omega(t),(\mathbf{I}+\mathbf{K})(\omega-\omega_0)) = \int_0^\infty p\frac{\varepsilon(p)\overline{\omega}_0(p)\overline{\omega}(p)}{J_1^2(p\gamma)+Y_1^2(p\gamma)}\,\mathrm{d}p.\end{aligned}$$

Now the Parseval theorem coupled with the inequalities $2xy \leqslant x^2+y^2$ and (41) gives

$$|\,W-\widetilde{W}\,| \leqslant \frac{\varepsilon}{2}\left[W+\frac{1}{1-\varepsilon}\widetilde{W}\right].$$

Upon using this estimate we achieve the errors for the punch energy

$$\frac{|\,W-\widetilde{W}\,|}{W} \leqslant \frac{\varepsilon(2-\varepsilon)}{2-3\varepsilon}, \quad \frac{|\,W-\widetilde{W}\,|}{\widetilde{W}} \leqslant \frac{\varepsilon}{1-\varepsilon}. \tag{42}$$

One might see that the algorithm remains valid if we take some $\widetilde{\gamma}$ instead of γ and any even rational approximation instead of the approximation (31).

Note that approximations similar to (31) are often used in the Russian literature for equations containing kernels of integral transforms which arise from the Sturm–Liouville problem (see, for example, Aizikovich and Trubchik, 1989). Their approach is based on explicit representations of the solutions in the form of finite combinations of the functions that obey the appropriate differential equation.

3. Case *n* = 1

A solution for $n = 1$ is constructed in a manner which is analogous to the manner suggested for the axially symmetric problem. The discontinuous integral

$$\int_0^\infty p\chi_{1,0}^{\gamma}(p,t)J_1(pr)\,\mathrm{d}p - \frac{2}{\pi\gamma r} = \begin{cases}\dfrac{2(r^2+\gamma^2-t^2)}{\pi\gamma r\sqrt{((r+\gamma)^2-t^2)((r-\gamma)^2-t^2)}}, & 0<r<t-\gamma,\\ 0, & r>t-\gamma,\end{cases} \tag{43}$$

evaluated by means of the integrals from the handbook by Prudnikov et al. (1986) is employed in place of (9). Its inversion gives

$$\mathbf{R}_1[J_1(pr)] = p\chi_{1,1}^{\gamma}(p,t), \tag{44}$$

$$\mathbf{R}_1[\cdot] = -\frac{2}{\pi\gamma}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^{t-\gamma}(\cdot)\frac{(r^2+\gamma^2-t^2)\mathrm{d}r}{\sqrt{(t^2-(r+\gamma)^2)(t^2-(r-\gamma)^2)}}. \tag{45}$$

These relations and the representation

$$\frac{2R}{\pi\gamma}A_1(p) = p\int_\gamma^{1+\gamma} s\psi(s)\chi_{1,1}^{\gamma}(p,s)\,\mathrm{d}s = p\overline{\psi}(p) \tag{46}$$

enable us to derive the Fredholm integral equation of the second kind

$$(\mathbf{I}+\mathbf{S})\psi = h(t), \quad \gamma \leqslant t \leqslant 1+\gamma, \tag{47}$$

$$\mathbf{S}\psi = \int_\gamma^{1+\gamma} s\psi(s)S(t,s)\mathrm{d}s, \quad h(t) = \mathbf{R}_1[u_1(r)], \tag{48}$$

$$S(t,s) = \int_0^\infty p[L(p)-1]\frac{\chi_{1,1}^{\gamma}(p,t)\chi_{1,1}^{\gamma}(p,t)}{J_1^2(p\gamma)+Y_1^2(p\gamma)}\mathrm{d}p. \tag{49}$$

The corresponding contact pressure can be found from the expression

$$Rp_1(r) = \frac{(1+2\gamma-r^2)(1+\gamma)\psi(1+\gamma)}{r\sqrt{(1-r^2)[(1+2\gamma)^2-r^2]}} + \int_{r+\gamma}^{1+\gamma}\frac{(r^2+\gamma^2-s^2)\mathrm{d}[s\psi(s)]}{r\sqrt{(s^2-(r+\gamma)^2)(s^2-(r-\gamma)^2)}} \tag{50}$$

that gives the stress intensity factor

$$\lim_{r\to 1} p_1(r)\sqrt{1-r^2} = \frac{\sqrt{\gamma+\gamma^2}}{R}\psi(1+\gamma). \tag{51}$$

The condition (4) becomes

$$2\pi R^2\mathbf{Re}\int_\gamma^{1+\gamma}\psi(s)(s^2-\gamma^2)\,\mathrm{d}s = \mathscr{P}d. \tag{52}$$

The work W_1 done by the contact pressure $p_1(r)$ in causing the conjugate punch displacement $u_1^*(r)$ is

$$W_1 = \frac{\pi^2(1-v_0)\gamma R}{G_0}\int_\gamma^{1+\gamma} s\psi(s)h^*(s)\,\mathrm{d}s. \tag{53}$$

It is readily seen that $m \leqslant \|\mathbf{I}+\mathbf{S}\| \leqslant M$, where numbers m and M are the same that were defined above in the axially symmetric case. This involves that the numerical methods pointed out above for Eq. (16) are also suitable for Eq. (47). The estimate for the norm of the integral operator remains the same as well: $\|\mathbf{S}\| \leqslant \sup|L(p)-1|$.

A slight correction permits to adopt the above method of the approximate reduction to the system of the algebraic equations. We take a solution of the form

$$\overline{\psi}(p) = \frac{1}{\widetilde{L}(p)}\int_\gamma^{1+\gamma} s\phi(s)\chi_{1,1}^{\gamma}(p,s)\,\mathrm{d}s - \frac{J_{l+1}(p)}{p^{l+1}}\sum_{k=1}^{N}\frac{y_k}{p^2+a_k^2}, \tag{54}$$

where $\phi(s)$ is a new unknown function, y_k are undetermined numbers, l is any positive number and $\widetilde{L}(p)$ is defined by (31).

The substitution (54) coupled with the approximation $L(p) \approx \widetilde{L}(p)$ leads to the equations

$$\phi(t) = h(t) + \sum_{m=0}^{N} y_m \phi_m(t), \quad \sum_{m=1}^{N} y_m c_{mk} = -h_k, \quad k = 1, 2, \ldots, N, \tag{55}$$

$$\phi_m(t) = \int_0^\infty \frac{p^{-l}\widetilde{L}(p) J_{l+1}(p) \chi_{1,1}^{\gamma}(p,t)\,\mathrm{d}p}{(p^2 + a_m^2)[J_1^2(p\gamma) + Y_1^2(p\gamma)]}, \quad h_k = \int_\gamma^{1+\gamma} h(s) W_{k,1}(s) s\,\mathrm{d}s,$$

$$c_{m,k} = \frac{\pi}{2} \sum_{r=1}^{N} \frac{(1+\gamma) U_{r,k} I_{l+1}(b_r) b_r^{-l-1} \prod_{l=1}^{N} (b_r^2 - a_l^2)}{(b_r^2 - a_k^2)(b_r^2 - a_m^2) K_1(b_r\gamma) \prod_{\substack{l=1 \\ l \neq r}}^{N} (b_r^2 - b_l^2)^{-1}}. \tag{56}$$

4. Indentation of a flat punch

For an inclined circular flat punch,

$$\begin{aligned} &u_0(r) = G_0 c/(1 - v_0) = w_0, \quad 2u_1(r) = c_1 r R G_0/(1 - v_0) = w_1 r, \\ &p_{-1}(r) = p_1(r). \end{aligned} \tag{57}$$

In this case, the right parts of the integral equations for the auxiliary functions can be evaluated explicitly:

$$\begin{aligned} g(t) &= w_0 \mathbf{R}_0[1] = w_0 \lim_{p \to 0} \mathbf{R}_0[J_0(pr)] = \frac{2w_0}{\pi\gamma}, \\ h(t) &= w_1 \mathbf{R}_1\left[\frac{r}{2}\right] = w_1 \lim_{p \to 0} \frac{1}{p} \mathbf{R}_1[J_1(pr)] = \frac{w_1}{\pi t \gamma}(t^2 - \gamma^2). \end{aligned}$$

We will solve Eq. (16) with the simplest and weakest version $i = 1$ of the algorithm (29):

$$\begin{aligned} &\omega_{k+1}(t) = \left(\frac{M + m - 2}{M + m}\mathbf{I} - \frac{2}{M + m}\mathbf{K}\right)\omega_k(t) + \frac{2}{M + m} g(t), \\ &\|\omega(t) - \omega_k(t)\| \leqslant \|\omega_0(t)\| \frac{q^{k+1}}{1 - q}, \quad q = \frac{M - m}{M + m}. \end{aligned}$$

Taking $\omega_0(t) = 2w_0/\pi\gamma$ and making one iteration, we obtain the approximate solution

$$\frac{\pi\gamma}{2w_0}\omega(t) \approx 1 - \frac{2(1+\gamma)}{M + m} \int_0^\infty [L(p) - 1] \frac{\chi_{1,1}^{\gamma}(p, 1+\gamma)\chi_{1,0}^{\gamma}(p,t)}{J_1^2(p\gamma) + Y_1^2(p\gamma)}\,\mathrm{d}p \tag{58}$$

whose absolute error is less than $2w_0\theta/\pi\gamma$, $\theta = (M - m)^2/[2m(M + m)]$.

An approximate solution of Eq. (47) having the error $w_1\theta/\pi\gamma$ can be derived in an analogous manner

$$\begin{aligned} &\frac{2\pi\gamma}{w_1}\psi(t) \approx \frac{1}{t}(t^2 - \gamma^2) - \frac{2}{M + m} \int_0^\infty [L(p) - 1] \frac{\chi_{1,1}^{\gamma}(p,t)\chi(p,\gamma)}{J_1^2(p\gamma) + Y_1^2(p\gamma)}\,\mathrm{d}p, \\ &\chi(p,\gamma) = (1+\gamma)^2 \chi_{1,2}^{\gamma}(p, 1+\gamma) + \gamma^2 \chi_{1,0}^{\gamma}(p, 1+\gamma). \end{aligned} \tag{59}$$

The approximate formulas connecting the displacements of the punch with the imbedding force now follow from the relations (21) and (52)

$$\frac{(1-\nu_0)\mathscr{P}}{RG_0c} \approx 4 + \frac{2}{\gamma} - \frac{8(1+\gamma)^2}{\gamma(M+m)} \int_0^\infty \frac{[L(p)-1]\left[\chi_{1,1}^{\gamma}(p,1+\gamma)\right]^2}{p\left[J_1^2(p\gamma)+Y_1^2(p\gamma)\right]} \mathrm{d}p, \tag{60}$$

$$\frac{2(1-\nu_0)\mathscr{P}d}{R^3G_0c_1} \approx 4 + \frac{1}{\gamma} + 2\gamma - 4\gamma^2 + 4\gamma^3 \ln\frac{1+\gamma}{\gamma} - \frac{8}{\gamma(M+m)} \int_0^\infty \frac{[L(p)-1]\chi^2(p,\gamma)}{p[J_1^2(p\gamma)+Y_1^2(p\gamma)]} \mathrm{d}p. \tag{61}$$

Of course, the above approximations are expected to be true as θ is small. Work out them for a homogeneous layer.

For the frictionless homogeneous layer (problem A),

$$f(p) = \frac{\cosh 2\lambda p - 1}{\sinh 2\lambda p + 2\lambda p}. \tag{62}$$

In this case $\gamma = \lambda/\pi$ and calculations yield: $m = 1$, $M = 1.284238$, $q = 1.2443 \times 10^{-2}$, $\theta = 1.76845 \times 10^{-2}$.

For the homogeneous layer which is bonded to the rigid base (problem B), we have

$$f(p) = \frac{2\kappa \sinh 2\lambda p - 4\lambda p}{2\kappa \cosh 2\lambda p + 4\lambda^2 p^2 + \kappa^2 + 1}, \quad \kappa = 3 - 4\nu. \tag{63}$$

The results of calculations are given in the Table 1, $\gamma\pi = \lambda(1-2\nu)/(1-\nu)^2$.

It is seen that one can employ the above approximate solutions for all λ at least if $\nu \leqslant 0.4$. We note that in the most of applications, such as geophysics, building and mechanical designing, and so forth, it is accepted to use $\nu = 0.25$ or 0.3. The estimates for the error θ deteriorate as the Poisson ratio ν approaches 1/2. But we remind that these estimates are very rough. In many cases, the algorithm remains efficient as ν close by 1/2 if some more accurate bounds of the eigenvalues is taken instead of M and m. So, it can be readily found that for $\lambda \gg 1$ the smallest upper bound of the spectrum is $1 + O(1/\lambda)$. The limiting case of the incompressible medium is peculiar since $\gamma \equiv 0$ for $\nu = 1/2$ and the lower bound $m = 0$. Moreover, if $\lambda \to 0$, then $\|\mathbf{I}+\mathbf{K}\| \to 0$, $\|\mathbf{I}+\mathbf{S}\| \to 0$ under any choice $\gamma = O(\lambda^c)$, $c < 3/2$, and the operators $\mathbf{I}+\mathbf{K}$, $\mathbf{I}+\mathbf{S}$ become unbounded as $\gamma = O(\lambda^c)$, $c \geqslant 3/2$. Consequently, for the bonded thin incompressible layer the algorithm is not effective.

Asymptotic analysis of integrals based on the Mellin transform technique (Bleistein and Handelsman, 1986) enables us to derive from the relations (60), (61), (58) and (59) very simple asymptotic formulas:

$$\widetilde{\mathscr{P}} = \frac{(1-\nu_0)\mathscr{P}}{RG_0c} = 4 + \frac{2-4D}{\gamma} + O\left(\frac{1}{\gamma^2}\right), \quad \lambda \to \infty, \tag{64}$$

$$\mathscr{M} = \frac{2(1-\nu_0)\mathscr{P}d}{R^3G_0c_1} = \frac{16}{3} + O\left(\frac{1}{\gamma^2}\right), \quad \lambda \to \infty, \tag{65}$$

Table 1
Results of calculations for the problem B

ν	m	M	$q = \frac{M-m}{M+m}$	θ
0	1	1.253756	0.1126	1.429×10^{-2}
0.25	1	1.273151	0.1202	1.641×10^{-2}
0.3	1	1.311557	0.1348	2.1×10^{-2}
0.4	1	1.587134	0.2269	6.662×10^{-2}
0.45	1	2.2816	0.3905	0.25026
0.49	1	8.5522	0.7906	2.98548
0.495	1	16.5526	0.8861	6.89024

and

$$\widetilde{\mathscr{P}} = \frac{2}{\gamma} + 4 - (1+\gamma)(\Gamma - \Gamma_0\delta) + B_0(\delta)\delta^2, \quad \lambda \to 0, \tag{66}$$

$$\mathscr{M} = \frac{1}{\gamma} + 4 - \Gamma + 2\gamma - (3\Gamma - 2\Gamma_0)\delta + B_1(\delta)\delta^2, \quad \lambda \to 0, \tag{67}$$

where $\delta = \gamma/(1+\gamma)$, $B_n(\delta) = O(1)$ as $\lambda \to 0$,

$$D = \frac{8}{\pi^2(M+m)} \int_0^\infty \frac{L\left(\frac{p}{\gamma}\right) - 1}{p[J_1^2(p) + Y_1^2(p)]} \mathrm{d}p,$$
$$\Gamma = \frac{8}{\pi(M+m)} \int_0^\infty \left[L\left(\frac{p}{\gamma}\right) - 1\right] \frac{\mathrm{d}p}{p^2}, \quad \Gamma_0 = \frac{4}{M+m}.$$

For the problem A, $D = 0.26266$, $\Gamma = 1.64456$, $\Gamma_0 = 1.75113$. For the problem B as $v = 0.3$, $D = 0.23554$, $\Gamma = 1.95395$, $\Gamma_0 = 1.73044$. The discrepancies of (64) and (65) from the exact asymptotic expansions (Vorovich et al., 1974)

$$\widetilde{\mathscr{P}} = 4\sum_{k=0}^{4}\left(\frac{a_0}{\lambda}\right)^k - \frac{20.064}{3\pi\lambda^3}\left(1 + \frac{4a_0}{\pi\lambda}\right), \quad a_0 = 1.377, \tag{68}$$

$$\mathscr{M} = \frac{16}{3}\left(1 + \frac{5.016}{3\pi\lambda^3}\right), \tag{69}$$

are <1% for $\lambda \geqslant 4$.

Choosing $B_0(\delta) = 2.76$, $B_1(\delta) = 10.5–28.5\delta^2$ for the problem A and $B_0(\delta) = 5.22$, $B_1(\delta) = 11.8–38\delta^2$ for the problem B, we obtain certain simple approximate formulas. Results for $\widetilde{\mathscr{P}}$ are summarized in Table 2, where the columns I and II give computations by formulas (60) and (66), respectively. The column III contains the values of $\widetilde{\mathscr{P}}$ which have been obtained as $\lambda \leqslant 2$ in Ufliand (1967) with the Cooke–Lebedev method by numerical analysis of the Fredholm equation of the second kind; for $\lambda = 4$ this one is found from the exact asymptotic formula (68). Table 3 for (61) and (67) has the same structure. We observe that our expressions are almost exact as $\lambda \leqslant 4$ and overlap the values (64) and (65) at $\lambda = 4$.

The asymptotic expansion for the stress intensity factor is given by the relation

$$\lim_{r\to 1}[p_0(r) + 2p_1(r)\cos\theta]\sqrt{1-r^2} = \frac{2G_0}{\pi(1-v_0)R}[c\sigma_0 + c_1\sigma_1 R\cos\theta], \tag{70}$$

$$\sqrt{\delta}\sigma_0 = \begin{cases} 1 - \left(1 - \frac{1}{2\gamma}\right)\frac{1}{\gamma}D + O\left(\frac{1}{\gamma^2}\right), & \lambda \to \infty, \\ 1 - \frac{2-\Gamma_0}{8}\delta + O(\delta^2), & \lambda \to 0, \end{cases} \tag{71}$$

Table 2
$\widetilde{\mathscr{P}} = \frac{(1-v_0)\mathscr{P}}{RG_0c}$

	A			B		
λ	I	II	III	I	II	III
1/3	21.219	21.242	21.04	25.142	25.152	24.80
1/2	14.945	14.991	14.88	17.468	17.482	17.36
1	8.7414	8.834	8.80	9.882	9.912	9.92
2	5.972	5.983	6.04	6.419	6.397	6.48
4	4.882	4.928	4.884	5.059	5.115	5.077

Table 3
$\mathscr{M} = \frac{2(1-\nu_0)\mathscr{P}d}{R^3 G_0 c_1}$

	A			B		
λ	I	II	III	I	II	III
1/3	11.885	11.950	11.888	13.683	13.734	13.632
1/2	8.848	8.950	8.944	9.950	10.007	10.016
1	6.233	6.305	6.352	6.601	6.613	6.752
2	5.523	5.587	5.552	5.601	5.571	5.632
4	5.381	5.379	5.361	5.402	5.408	5.378

$$\sqrt{\delta}\sigma_1 = \begin{cases} 1 - O\left(\dfrac{1}{\gamma^2}\right), & \lambda \to \infty, \\ 1 + \delta - \dfrac{6 - 3\Gamma_0}{8}\gamma + \dfrac{2}{M+m}\delta^2 + Q(\delta), & \lambda \to 0, \end{cases} \tag{72}$$

where $Q(\delta) = O(\delta^3)$.

The results of calculations for σ_0 and σ_1 are represented in Tables 4 and 5, which are of the same structure as Table 2. It is taken $Q(\delta) = C\delta^3$ in the corresponding asymptotic expansions: $C = 2.6$ for the problem A and $C = 3.63$ for the problem B as $\nu = 0.3$. We again see that the approximate formulas are remarkable accurate.

We accentuate that the leading terms of all asymptotic expansions obtained above, both $\lambda \to \infty$ and $\lambda \to 0$, coincide with the exact leading terms of the asymptotic expansions (Vorovich et al., 1974). One might guess that the leading terms of the asymptotic expansions as $\gamma \ll 1$ for the inhomogeneous stratum have the same form:

$$\mathscr{P} \sim \frac{2G_0Rc}{\gamma(1-\nu_0)}, \quad \mathscr{P}d \sim \frac{G_0R^3c_1}{2\gamma(1-\nu_0)}, \tag{73}$$

Table 4
The stress intensity factor σ_0

	A			B		
λ	I	II	III	I	II	III
1/3	3.165	3.165	3.142	3.472	3.472	3.439
1/2	2.622	2.623	2.608	2.864	2.566	2.845
1	1.923	1.934	1.915	2.082	2.090	2.071
2	1.456	1.4752	1.460	1.548	1.566	1.551
4	1.215	1.182	1.214	1.257	1.224	1.258

Table 5
The stress intensity factor σ_1

	A			B		
λ	I	II	III	I	II	III
1/3	1.678	1.680	1.672	1.812	1.813	1.802
1/2	1.429	1.434	1.426	1.528	1.529	1.524
1	1.144	1.153	1.148	1.194	1.194	1.198
2	1.033	1.028	1.032	1.047	1.033	1.047
4	1.008	1.006	1.005	1.011	1.018	1.008

and

$$\lim_{r\to 1}[p_0(r)+2p_1(r)\cos\theta]\sqrt{1-r^2}\sim \frac{2G_0}{\pi(1-\nu_0)R\sqrt{\gamma}}[c+c_1R\cos\theta]\sim \frac{\mathscr{P}\sqrt{\gamma}}{\pi R^2}\left[1+4\frac{d}{R}\cos\theta\right], \tag{74}$$

under the condition that the coefficients of the expansion of $f(p/\gamma)$ in ascending powers of p are not great numbers. This statement should be, however, rigorously proved.

The generalization of the relations (73) for the punch having the three-dimensional profile $v(\rho,\theta)$, $\rho = Rr$, can be easily derived

$$\frac{\gamma(1-\nu_0)}{2G_0R}\mathscr{P}\sim c-\frac{1}{\pi R^2}\int_0^R\int_0^{2\pi}v(\rho,\theta)\rho\,\mathrm{d}\rho\,\mathrm{d}\theta, \tag{75}$$

$$\frac{2\gamma(1-\nu_0)}{G_0R^3}\mathscr{P}d\sim c_1-\frac{4}{\pi R^4}\int_0^R\int_0^{2\pi}v(\rho,\theta)\rho^2\cos\theta\,\mathrm{d}\rho\,\mathrm{d}\theta. \tag{76}$$

Note that the relations (75) and (76) are exactly the solution for a Winkler foundation of stiffness $k = 2G_0/\pi(1-\nu_0)\gamma R$ indented by a rigid indenter. One might find from (19) and (50) that, at least for the harmonics $n = 0, 1$, the leading terms of the contact pressures obey Winkler's law $p_n(r) = ku_n(r)$ everywhere excepting very narrow neighborhoods of the edge and points of discontinuity of the derivatives $u_n'(r)$. In the case of the homogeneous layer and a smooth punch, this confirms Barber's approximate theory (Barber, 1990). For example, if the thin layer is indented by a rigid sphere of radius R_0, then $p(r) = k(c-r^2/2R_0)$. The condition $p(R) = 0$ now yields $2R_0c = R^2$ and

$$p(r)=\frac{k}{2R_0}(R^2-r^2), \tag{77}$$

where the contact radius is determined from (75): $2\pi kR^3 = 4R_0\mathscr{P}$. For the homogeneous layer, this asymptotic solution is well known (Vorovich et al., 1974; Jaffar, 1989).

We also see from the foregoing that the concept of "a thin layer" is not only geometric but depend via γ upon elastic properties and stratification of the medium as well. In particular, the foregoing theory does not work for the incompressible stratum bonded to the rigid substrate. The results for the homogeneous layer (Johnson, 1985; Barber, 1990; Jaffar, 1989; Alexandrov, 2003) predict a dramatic change of the asymptotic behavior of the thin stratum in this limit. Author's paper (Malits, 2004) extents the technique developed here to this case.

In conclusion, we emphasize the essential difference between Eqs. (16) and (47) and equations obtained with the Cooke–Lebedev method (or, equivalently, with the orthogonal polynomials method) whose operators in the studied problem tend to zero as $\gamma\to 0$.

References

Aizikovich, S.M., Trubchik, I.S., 1989. Asymptotic properties of the approximate solution of a class of dual integral equations. Soviet Phys. Dokl. 34 (7), 658–660.

Alexandrov, V.M., 2003. Asymptotic solution of the axisymmetric contact problem for an elastic layer of incompressible material. J. Appl. Math. Mech. (PMM) 67 (4), 589–593.

Barber, J.R., 1990. Contact problem for the thin elastic layer. Int. J. Mech. Sci. 31 (2), 129–132.

Bleistein, N., Handelsman, R.A., 1986. Asymptotic Expansions of Integrals. Dover, New York.

Cooke, J., 1956. A solution of Tranter's dual integral equations problem. Quart. J. Mech. Appl. Math. 9 (1), 103–110.

Jaffar, M.J., 1989. Asymptotic behaviour of thin elastic layers bonded and unbonded to a rigid foundation. Int. J. Mech. Sci. 32 (3), 229–235.

Johnson, K.L., 1985. Contact Mechanics. Cambridge University Press, Cambridge.

Krasnosel'skii, M.A., Vainikko, G.M., Zabreiko, R.P., Ruticki, Ya.B., Stet'senko, V.Ya., 1972. Approximate Solution of Operator Equations. Aspen. Publish. Inc.

Lebedev, N.N., 1957. The distribution of electricity on a thin parabolic segment. Doklady Akademii Nauk SSSR 114 (3), 513–516.

Malits, P., 2004. Indentation of an incompressible layer by a circular indenter. IJSS, submitted for publication.

Prudnikov, A.P., Brychkov, Yu.A., Marichev, O.I., 1986. Integrals and Series, vol. 2. Gordon and Breach Science Publishers, Scripta Technica Inc., New York.

Shevlyakov, Yu.A., 1977. Matrix Algorithms in Theory of Elasticity of Inhomogeneous Media. Vischa Skola, Kiev-Odessa, in Russian.

Titchmarsh, E.C., 1924. Eigenfunction Expansions Associated with Second Order Differential Equations. Clarendon Press, Oxford.

Ufliand, Ya.S., 1967. Integral Transforms in Problems of Theory of Elasticity. Nauka, Leningrad. English trans. Survey of articles on Appl. of Integral Transforms in Problems of Theory of Elasticity, North Carolina State Univ., Appl. Math. Res. Grp. File No PSR-24/6,1965.

Vorovich, I.I., Alexandrov, V.M., Babeshko, V.A., 1974. Non-classical Mixed Boundary Value Problems of Elasticity. Nauka, Moskow, in Russian.